Liouville theory without an action

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Abstract

We show that the crossing symmetry of the four-point function in the Liouville conformal field theory on the sphere contains more information than what was hitherto considered. Under certain assumptions, it provides the special structure constants that were previously computed perturbatively and allows to solve the theory without using the Liouville interaction.

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1 Introduction

The constraints imposed by symmetries on a physical theory achieve their most powerful realization when they completely determine its dynamics. A typical example of this occurs in rational conformal field theories (CFT) in two dimensions [1, 2], where the dynamics of the local fields can be solved without using any action, even though in many cases (as in WZW models for compact groups or unitary minimal models) the theories have well-defined Lagrangian realizations.

Dynamical constraints on a two-dimensional CFT follow from the existence of degenerate representations of the chiral algebra and the crossing symmetry, i.e., associativity of the operator product expansion [1]. In the last years significant progress was made in implementing these constraints in interacting non-rational CFTs, i.e., theories with a continuous set of primaries (for reviews see [3, 4, 5, 6]). In this note we will focus on Liouville theory, which is the simplest and best understood example of this class of models, and is important in the study of non-critical strings [7]. Extending previous results, we will see that the conformal bootstrap program can be carried another step forward, allowing to compute the three-point function without using the Liouville interaction at all.

The Liouville theory is defined by the action

$$S_L[\phi] = \frac{1}{4\pi} \int d^2z \left[\partial\phi\bar{\partial}\phi + RQ\phi \right] + \mu \int d^2z \, e^{2b\phi} \,. \tag{1}$$

It is a CFT with central charge

$$c = 1 + 6Q^2 \,, \tag{2}$$

where

$$Q = b + \frac{1}{b} \,. \tag{3}$$

For $b \in \mathbb{R}$, we have $c \geq 25$. The observables $V_\alpha = e^{2\alpha\phi}$ have conformal dimension

$$\Delta_\alpha = \alpha(Q - \alpha) \,, \tag{4}$$

and normalizable states correspond to $\alpha = \frac{Q}{2} + i\mathbb{R}^+$ [8]. Since these do not belong to degenerate representations of the chiral algebra, not all the methods of [1] can be used. Early attempts to compute correlators resorted thus to more conventional path integral or operator quantizations methods, based on the action. In the path integral approach, for example, correlation functions are given by[1]

$$\langle e^{2\alpha_1\phi(z_1)} \dots e^{2\alpha_n\phi(z_n)} \rangle = \int \mathcal{D}\phi \prod_{r=1}^{n} e^{2\alpha_r\phi(z_r)} e^{-S_L[\phi]} \,. \tag{5}$$

It is useful to split the Liouville field into a zero mode and fluctuations around it as $\phi = \phi_0 + \tilde{\phi}$. Performing the formal integral over ϕ_0 in (5) yields [9]

$$\langle e^{2\alpha_1\phi(z_1)} \dots e^{2\alpha_n\phi(z_n)} \rangle = \frac{\Gamma(-s)}{2b} \langle \prod_{r=1}^{n} e^{2\alpha_r\tilde{\phi}(z_r)} \left(\mu \int d^2z \, e^{2b\tilde{\phi}(z)} \right)^s \rangle_{free} \tag{6}$$

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[1] We omit the explicit anti-holomorphic dependence of the fields unless needed.

where

$$s = \frac{Q - \sum_{r=1}^{n} \alpha_r}{b} \, . \tag{7}$$

The resulting expression makes sense when s is a non-negative integer. In this case the factor $\Gamma(-s)$ gives a pole associated to the non-compactness of the target space, and a free field computation gives the residue. In particular, when $\sum_{r=1}^{n} \alpha_r = Q$, we have $s = 0$ and no insertions of the Liouville interaction are required. For two vertices, this suggests adopting the state $\langle Q - \alpha |$ as the BPZ conjugate of $|\alpha\rangle$, since V_α and $V_{Q-\alpha}$ have the same conformal dimension Δ_α in (4), and

$$\langle Q - \alpha | \alpha \rangle = \lim_{z \to \infty} |z|^{2\Delta_\alpha} \langle e^{2(Q-\alpha)\phi(z,\bar{z})} e^{2\alpha\phi(0,0)} \rangle = 1 \, , \tag{8}$$

up to a divergent factor associated to the non-compactness of the zero mode.

These results led the authors of [10, 11, 12] to propose the following expression for the three point function

$$\langle V_{\alpha_1}(z_1) V_{\alpha_2}(z_2) V_{\alpha_3}(z_3) \rangle = \frac{C(\alpha_1, \alpha_2, \alpha_3)}{|z_{12}|^{2\Delta_1 + 2\Delta_2 - 2\Delta_3} |z_{23}|^{2\Delta_2 + 2\Delta_3 - 2\Delta_1} |z_{31}|^{2\Delta_3 + 2\Delta_1 - 2\Delta_2}} \, , \tag{9}$$

with

$$C(\alpha_1, \alpha_2, \alpha_3) = \left[\pi \mu \gamma(b^2) b^{2-2b^2} \right]^{\frac{(Q-2\tilde{\alpha})}{b}} \frac{\Upsilon'(0)}{\Upsilon(2\tilde{\alpha} - Q)} \prod_{r=1}^{3} \frac{\Upsilon(2\alpha_r)}{\Upsilon(2\tilde{\alpha}_r)} \, . \tag{10}$$

We have defined

$$\tilde{\alpha} = \frac{1}{2}(\alpha_1 + \alpha_2 + \alpha_3) \qquad \tilde{\alpha}_r = \tilde{\alpha} - \alpha_r \, , \tag{11}$$

$$\gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)} \, , \tag{12}$$

and the definition and properties of the function $\Upsilon(x)$ are recalled in Appendix A.

Eq.(10), so-called DOZZ formula, has poles at the expected values of s with the correct residues, and has passed several consistency tests[2]. But the full power of the conformal symmetry remained hidden from this important result until it was shown in [13] how the DOZZ formula fits into the framework of the conformal bootstrap of [1]. This approach, dubbed the Teschner trick, led to the computation of other bulk and boundary quantities in Liouville theory and in other non-rational CFTs, such as $N = 1, 2$ Liouville and the H_3^+ WZW model (see [6] for full references).

The method of [13], which we review below, still used the Liouville Lagrangian, but reduced its role to a minimum. To obtain the three-point function, for example, one only needs to compute a special structure constant $C_-(\alpha)$ appearing in the fusion of the degenerate Virasoro primary $V_{-b/2}$ with a generic primary V_α:

$$V_{-b/2} V_\alpha = C_+(\alpha) \left[V_{\alpha-b/2} \right] + C_-(\alpha) \left[V_{\alpha+b/2} \right] \, . \tag{13}$$

[2]In fact, it has more poles than expected. This is due to the $b \leftrightarrow b^{-1}$ duality present in the full quantum theory.

As follows from (7) and (8), $C_-(\alpha)$ can be obtained perturbatively using just one insertion of the Liouville interaction,

$$
\begin{aligned}
C_-(\alpha) &= \langle V_{Q-\alpha-\frac{b}{2}}(\infty)V_{-\frac{b}{2}}(1)V_\alpha(0)\rangle \\
&= -\mu \int d^2x \langle V_{Q-\alpha-\frac{b}{2}}(\infty)e^{2b\phi}(x)V_{-\frac{b}{2}}(1)V_\alpha(0)\rangle_{free} \\
&= -\mu \int d^2x |x|^{-4\alpha b}|x-1|^{2b^2} \\
&= -\pi\mu \frac{\gamma(-1+2\alpha b - b^2)}{\gamma(2\alpha b)\gamma(-b^2)}\,,
\end{aligned}
\tag{14}
$$

where in the last line we have used the formula (A.5). The second special structure constant is

$$
C_+(\alpha) = \langle V_{Q-\alpha+b/2}(\infty)V_\alpha(1)V_{-b/2}(0)\rangle = 1\,,
\tag{15}
$$

since, according to (7), we need here no insertions of the Liouville interaction.[3]

The main result of this note is that, assuming that (8) and (15) hold, we obtain the special structure constant $C_-(\alpha)$ from the conformal bootstrap, thus allowing to completely solve the theory without ever using the Liouville interaction.

2 Exploiting the bootstrap

The bootstrap program for two-dimensional CFTs consists in determining the three-point functions from the constraints imposed by the crossing symmetry of the four-point functions [1]. The original approach to this program, successfully applied in [1, 14] for the minimal models, required the construction of monodromy invariant combinations of conformal blocks for Virasoro degenerate fields of arbitrarily high order.

The trick of Teschner [13] is an alternative implementation of the bootstrap idea, appropriate to compute correlators of non-degenenerate primaries. Instead of yielding the three-point functions themselves, the method gives difference equations for them. The latter must then be solved to obtain the final answer. Compared to the original bootstrap the method is very efficient, because for *any* three-point function one only needs the conformal blocks of the first Virasoro degenerate fields, which are given by standard hypergeometric functions.

Let us start by recapitulating the argument in [13]. The global conformal symmetry fixes the form of a four-point function to be

$$
\begin{aligned}
\langle V_{\alpha_4}(z_4)V_{\alpha_3}(z_3)V_{\alpha_2}(z_2)V_{\alpha_1}(z_1)\rangle &= G_{\alpha_4\alpha_3\alpha_2\alpha_1}(\eta,\bar\eta) \\
&\times |z_{24}|^{-4\Delta_2}|z_{14}|^{2(\Delta_2+\Delta_3-\Delta_1-\Delta_4)}|z_{34}|^{2(\Delta_1+\Delta_2-\Delta_3-\Delta_4)}|z_{13}|^{2(\Delta_4-\Delta_1-\Delta_2-\Delta_3)}
\end{aligned}
\tag{16}
$$

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[3]There is a slight abuse of notation here. Following (6)-(7), the r.h.s. of (14) and (15) are expected to have a pole, and they should be read as defining the corresponding residue. The same holds for similar expressions below. We will also evaluate quotients of $C(\alpha_1, \alpha_2, \alpha_3)$ at divergent values. In these cases the poles will cancel and we get the quotient of the residues.

where

$$\eta = \frac{z_{12}z_{34}}{z_{13}z_{24}} \qquad \bar{\eta} = \frac{\bar{z}_{12}\bar{z}_{34}}{\bar{z}_{13}\bar{z}_{24}} \, . \tag{17}$$

Among the non-normalizable states, the pair

$$\alpha_{2,1} = -\frac{b}{2} \qquad \alpha_{1,2} = -\frac{1}{2b} \, , \tag{18}$$

corresponds to degenerate states with a singular descendant at level 2. To obtain constraints on the three-point function $C(\alpha_1, \alpha_2, \alpha_3)$, we consider a four-point function with $\alpha_2 = \alpha_{2,1} = -b/2$. We are assuming that the values of α_i in the correlators can be analytically continued from the normalizable values to non-normalizable ones. The decoupling of the singular vector

$$\left[L_{-2} + \frac{1}{b^2} L_{-1}^2 \right] V_{-b/2} = 0 \, , \tag{19}$$

implies then that $G_{\alpha_4\alpha_3\alpha_2\alpha_1}(\eta, \bar{\eta})$ satisfies

$$\left[-\frac{1}{b^2} \frac{d^2}{d\eta^2} + \left(\frac{1}{\eta-1} + \frac{1}{\eta} \right) \frac{d}{d\eta} - \frac{\Delta_3}{(\eta-1)^2} - \frac{\Delta_1}{\eta^2} + \frac{\Delta_1 + \Delta_2 + \Delta_3 - \Delta_4}{\eta(\eta-1)} \right] G_{\alpha_4\alpha_3\alpha_2\alpha_1}(\eta, \bar{\eta}) \, , \tag{20}$$

and a similar anti-holomorphic equation. The solutions to this equation near $\eta \sim 0, 1, \infty$ go like $\eta^{\Delta_{\alpha_1 \pm b/2} - \Delta_1 - \Delta_2}$, $(1-\eta)^{\Delta_{\alpha_3 \pm b/2} - \Delta_3 - \Delta_2}$ and $(1/\eta)^{\Delta_{\alpha_4 \pm b/2} - \Delta_4 - \Delta_2}$, respectively. From this we see that the fusion of $V_{-b/2}$ with any primary V_α gives $V_{\alpha \pm b/2}$, as in (13)[4]. Also from (20) follows that the conformal blocks can be expressed through hypergeometric functions. Expanding around $\eta \sim 0$, we get

$$G_{\alpha_4\alpha_3\alpha_2\alpha_1}(\eta) = \sum_{s=+,-} C(\alpha_4, \alpha_3, \alpha_1 - sb/2) C_s(\alpha_1) \left| \mathcal{F}_s \begin{bmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{bmatrix}(\eta) \right|^2 \, , \tag{21}$$

where

$$C_\pm(\alpha) \equiv C(Q - \alpha \pm b/2, -b/2, \alpha) \tag{22}$$

are the special structure constants in (13). The conformal blocks are

$$\mathcal{F}_+ \begin{bmatrix} \alpha_3 & -b/2 \\ \alpha_4 & \alpha_1 \end{bmatrix}(\eta) = \eta^{\Delta_{\alpha_1} - b/2 - \Delta_1 - \Delta_2}(1-\eta)^{b\alpha_3} F(A, B; C; \eta) \, , \tag{23}$$

$$\mathcal{F}_- \begin{bmatrix} \alpha_3 & -b/2 \\ \alpha_4 & \alpha_1 \end{bmatrix}(\eta) = \eta^{\Delta_{\alpha_1} + b/2 - \Delta_1 - \Delta_2}(1-\eta)^{b\alpha_3} F(A - C + 1, B - C + 1; 2 - C; \eta) \, , \tag{24}$$

with

$$A = -1 + b(\alpha_1 + \alpha_3 + \alpha_4 - 3b/2) \, , \tag{25}$$
$$B = b(\alpha_1 + \alpha_3 - \alpha_4 - b/2) \, , \tag{26}$$
$$C = 2\alpha_1 b - b^2 \, . \tag{27}$$

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[4]Note that eq.(13) assumes that there is only one field for every conformal dimension (the vertices V_α and $V_{Q-\alpha}$ are related by the reflection coefficient $R(\alpha)$ as $V_\alpha = R(\alpha) V_{Q-\alpha}$, see [12]). This property would distinguish between Liouville theory and a direct sum of several Liouville CFTs with the same total central charge.

The interchange of $V_{\alpha_1}(z_1)$ and $V_{\alpha_4}(z_4)$ in (16) leads to the crossing symmetry relation

$$G_{\alpha_4\alpha_3\alpha_2\alpha_1}(\eta,\bar\eta) = |\eta|^{-4\Delta_2}G_{\alpha_1\alpha_3\alpha_2\alpha_4}(1/\eta,1/\bar\eta), \qquad (28)$$

which will lead to the crucial relation to exploit below. Now, the s-channel conformal blocks (23)-(24) can be expressed through the u-channel conformal blocks $\mathcal{F}_\pm\left[\begin{smallmatrix}\alpha_3\,\alpha_2\\\alpha_1\,\alpha_4\end{smallmatrix}\right](1/\eta,1/\bar\eta)$ by means of the standard identity

$$
\begin{aligned}
F(A,B;C;\eta) &= \frac{\Gamma(C)\Gamma(B-A)}{\Gamma(B)\Gamma(C-A)}(-\eta)^{-A}F(A,1-C+A;1-B+A;1/\eta)\\
&+ \frac{\Gamma(C)\Gamma(A-B)}{\Gamma(A)\Gamma(C-B)}(-\eta)^{-B}F(B,1-C+B;1-A+B;1/\eta).
\end{aligned}
\qquad (29)
$$

This leads to a fusing relation of the form

$$\mathcal{F}_s\left[\begin{smallmatrix}\alpha_3\,\alpha_2\\\alpha_4\,\alpha_1\end{smallmatrix}\right](\eta,\bar\eta) = \sum_{r=\pm} D_{sr}\mathcal{F}_r\left[\begin{smallmatrix}\alpha_3\,\alpha_2\\\alpha_1\,\alpha_4\end{smallmatrix}\right](1/\eta,1/\bar\eta). \qquad (30)$$

Inserting this transformation into (21) and defining $\mathcal{F}_\pm \equiv \mathcal{F}_\pm\left[\begin{smallmatrix}\alpha_3\,\alpha_2\\\alpha_1\,\alpha_4\end{smallmatrix}\right](1/\eta,1/\bar\eta)$, we see that (28) is equivalent to

$$
\begin{aligned}
& C(\alpha_4,\alpha_3,\alpha_1-b/2)\,C_+(\alpha_1)\ |D_{++}\mathcal{F}_+ + D_{+-}\mathcal{F}_-|^2 \\
+\ & C(\alpha_4,\alpha_3,\alpha_1+b/2)\,C_-(\alpha_1)\ |D_{-+}\mathcal{F}_+ + D_{--}\mathcal{F}_-|^2 \\
=\ & C(\alpha_1,\alpha_3,\alpha_4-b/2)\,C_+(\alpha_4)\ |\mathcal{F}_+|^2 + C(\alpha_1,\alpha_3,\alpha_4+b/2)\,C_-(\alpha_4)\,|\mathcal{F}_-|^2.
\end{aligned}
\qquad (31)
$$

This is the main equation to exploit. First note that by setting to zero the cross terms in the l.h.s. we get

$$
\begin{aligned}
\frac{C(\alpha_4,\alpha_3,\alpha_1+b/2)}{C(\alpha_4,\alpha_3,\alpha_1-b/2)} &= -\frac{C_+(\alpha_1)}{C_-(\alpha_1)}\frac{D_{++}\bar D_{+-}}{D_{-+}\bar D_{--}}, \qquad (32)\\
&= \frac{C_+(\alpha_1)}{C_-(\alpha_1)}\frac{\gamma(2\alpha_1 b - b^2)}{\gamma(2-2\alpha_1 b + b^2)}\times \\
&\times\ \frac{\gamma(b(\alpha_4+\alpha_3-\alpha_1)-b^2/2)}{\gamma(-1-3b^2/2+b(\alpha_1+\alpha_3+\alpha_4))}\,\frac{\gamma(1+b(\alpha_4-\alpha_3-\alpha_1)+b^2/2)}{\gamma(-b^2/2+b(\alpha_1-\alpha_3+\alpha_4))}.
\end{aligned}
$$

This is a difference equation for $C(\alpha_4,\alpha_3,\alpha_1)$. It depends on the ratio $C_+(\alpha_1)/C_-(\alpha_1)$, which can be obtained from a perturbative computation, as we did in (14) and (15).

But it turns out that the crossing symmetry relation (31) contains more information. A new equation can be obtained by plunging (32) back into (31) and considering the coefficients of $|\mathcal{F}_-|^2$. This gives

$$
\begin{aligned}
\frac{C_-(\alpha_4)C(\alpha_1,\alpha_3,\alpha_4+b/2)}{C_-(\alpha_1)C(\alpha_4,\alpha_3,\alpha_1+b/2)} &= |D_{--}|^2 - \frac{D_{+-}D_{-+}\bar D_{--}}{D_{++}}, \qquad (33)\\
&= \frac{\Gamma^2(2+b^2-2b\alpha_1)\,\Gamma^2(2b\alpha_4-1-b^2)}{\Gamma^2(b(\alpha_3+\alpha_4-\alpha_1-b/2))\,\Gamma^2(1+b^2/2+b(\alpha_4-\alpha_3-\alpha_1))}+ \\
+\ & \frac{\Gamma^2(-1+2b\alpha_4-b^2)\sin(\pi b(\alpha_1-\alpha_3+\alpha_4-b/2))\sin(\pi b(\alpha_1+\alpha_3+\alpha_4-b/2-Q))}{\pi\gamma(-1+2b\alpha_1-b^2)\gamma(b(\alpha_3+\alpha_4-\alpha_1-b/2))\gamma(1+b^2/2+b(\alpha_4-\alpha_3-\alpha_1))\sin(\pi b(2\alpha_1-b))}.
\end{aligned}
$$

5

This equation is the main results of this note. No new equations arise from the coefficients of $|\mathcal{F}_+|^2$ in (31).

The quantum Liouville theory is assumed to be invariant under the interchange $b \leftrightarrow 1/b$. Thus the degenerate primary $\alpha_{1,2} = -b^{-1}/2$ leads to a second pair of functional equations, obtained from (32) and (33) by replacing b with $1/b$. Notice that the decoupling equation (19) for the $V_{-b/2}$ state can be traced back to the classical Liouville equation of motion, but for $V_{-b^{-1}/2}$ the decoupling equation has no classical limit, and is an additional assumption of the quantum theory. This is natural in our abstract approach, where the classical limit or the action play no role and both decoupling equations stand on the same footing.

We will denote the special structure constants associated to $V_{-b^{-1}/2}$ as

$$\tilde{C}_\pm(\alpha) \equiv C(Q - \alpha \pm b^{-1}/2, -b^{-1}/2, \alpha). \tag{34}$$

Consistency check on the BPZ conjugation

Let us see first how eq.(33) gives a consistency check on the BPZ conjugation (8), by fixing the quantity[5]

$$N(\alpha) = C(\alpha, Q - \alpha, 0) = \langle V_\alpha V_{Q-\alpha} \rangle. \tag{35}$$

Consider (33) at $\alpha_1 = \alpha, \alpha_3 = 0, \alpha_4 = Q - \alpha - b/2$. For these values, the r.h.s. of eq.(33) becomes 1, and using $C_-(\alpha) = C_-(Q - \alpha - b/2)$, we get

$$N(\alpha + b/2) = N(\alpha). \tag{36}$$

This condition along with a similar one with $b \leftrightarrow 1/b$, implies that, for incommensurate b and $1/b$, $N(\alpha)$ is a constant for *any* normalization of the vertex operators. One usually assumes that the latter are rescaled so that

$$N(\alpha) = 1. \tag{37}$$

The special structure constants and the DOZZ formula

A relation between the special structure constants $C_\pm(\alpha)$ can also be obtained evaluating eq.(33) at $\alpha_1 = \alpha, \alpha_3 = Q - \alpha, \alpha_4 = -b/2$. This gives[6]

$$C_-(\alpha)C_+(\alpha + b/2) = -\frac{\pi\mu}{\gamma(-b^2)} \frac{\gamma(-1 - b^2 + 2b\alpha)}{\gamma(2b\alpha)}, \tag{38}$$

where we have used (37) and we have defined μ as

$$\frac{\pi\mu}{\gamma(-b^2)} \equiv \frac{C_-(-b/2)}{2\cos(\pi b^2)} \frac{\Gamma^2(-b^2)}{\Gamma^2(-1 - 2b^2)}. \tag{39}$$

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[5]As in the definition of $C_\pm(\alpha)$, this expression should be understood as the residue of a pole (see footnote 3).

[6]In [15] it was pointed out that if we evaluate eq.(32) at $\alpha_1 = \alpha_4 = \alpha, \alpha_3 = -b/2$, then we get a relation between $C_-(\alpha)C_+(\alpha + b/2)$ and $C_-(\alpha - b/2)C_+(\alpha)$, but this is not enough to fix $C_-(\alpha)C_+(\alpha + b/2)$. Here we obtain its precise value, up to the free constant μ.

The value of $C_-(-b/2)$ is a free parameter, and it is convenient to express it through μ.

In (38), an arbitrary election for $C_+(\alpha)$ determines the structure constant $C_-(\alpha)$. This is the most we can expect from self-consistency of the theory, since the structure constants will change under α-dependent rescalings of the vertex operators. Assuming that the vertex operators are normalized such that $C_+(\alpha) = 1$, then μ becomes the cosmological constant in the Liouville Lagrangian, and we get for $C_-(\alpha)$ in (38) precisely the perturbative result (14)[7]. Similar relations hold for the $\tilde{C}_\pm(\alpha)$ structure constants, and we denote the dual cosmological constant by $\tilde{\mu}$.

Plunging the value of $C_-(\alpha)$ from (38) into (32) we get the difference equation

$$
\begin{aligned}
\frac{C(\alpha_3, \alpha_2, \alpha_1 + b)}{C(\alpha_3, \alpha_2, \alpha_1)} &= -\frac{\gamma(-b^2)}{\mu\pi} \frac{\gamma(2\alpha_1 b)}{\gamma(1 - 2\alpha_1 b - b^2)} \\
&\times \frac{\gamma(b(\alpha_3 + \alpha_2 - \alpha_1 - b))}{\gamma(b(\alpha_1 + \alpha_2 + \alpha_3 - Q))} \frac{\gamma(1 + b(\alpha_3 - \alpha_2 - \alpha_1))}{\gamma(b(\alpha_1 - \alpha_2 + \alpha_3))},
\end{aligned} \tag{40}
$$

and similarly

$$
\begin{aligned}
\frac{C(\alpha_3, \alpha_2, \alpha_1 + b^{-1})}{C(\alpha_3, \alpha_2, \alpha_1)} &= -\frac{\gamma(-b^{-2})}{\tilde{\mu}\pi} \frac{\gamma(2\alpha_1 b^{-1})}{\gamma(1 - 2\alpha_1 b^{-1} - b^{-2})} \\
&\times \frac{\gamma(b^{-1}(\alpha_3 + \alpha_2 - \alpha_1 - 1/b))}{\gamma(b^{-1}(\alpha_1 + \alpha_2 + \alpha_3 - Q))} \frac{\gamma(1 + b^{-1}(\alpha_3 - \alpha_2 - \alpha_1))}{\gamma(b^{-1}(\alpha_1 - \alpha_2 + \alpha_3))}.
\end{aligned} \tag{41}
$$

These functional equations were first obtained by Teschner in [13]. Their solution is given by the DOZZ formula (10), and μ and its dual $\tilde{\mu}$ are related as

$$
\tilde{\mu}\pi\gamma(b^{-2}) = \left(\mu\pi\gamma(b^2)\right)^{1/b^2} . \tag{42}
$$

The solution is unique for incommensurable b and b^{-1}.

3 Conclusions

In this note we have reduced the dependence of the Liouville conformal field theory from its Lagrangian, by showing how the theory can be solved without using the Liouville interaction at all. The equations (8) and (15) are assumptions which follow naturally from the perturbative result (6), but do not involve the interaction. We believe our result will be useful to formulate for a purely constructive approach to Liouville theory, completely independent from a local Lagrangian.

Ackowledgements

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[7]An interesting point, which we shall not address here, is whether any arbitrary election for $C_+(\alpha)$ leads to a consistent theory, and when such an arbitrary election can be brought to $C_+(\alpha) = 1$ by rescaling the vertex operators.

A   Useful formulae

$$\gamma(x) \equiv \frac{\Gamma(x)}{\Gamma(1-x)} \tag{A.1}$$

$$\gamma(x) = \frac{1}{\gamma(1-x)} \tag{A.2}$$

$$\gamma(x+1) = -x^2\gamma(x) \tag{A.3}$$

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)} \tag{A.4}$$

$$\int_{\mathbb{R}^2} d^2x\, x^a \bar{x}^{\bar{a}}(1-x)^b(1-\bar{x})^{\bar{b}} = \pi\frac{\Gamma(1+a)}{\Gamma(-\bar{a})}\frac{\Gamma(1+b)}{\Gamma(-\bar{b})}\frac{\Gamma(-\bar{a}-\bar{b}-1)}{\Gamma(a+b+2)} \tag{A.5}$$

The integral is well defined only when $a - \bar{a}, b - \bar{b} \in Z$.

The function $\Upsilon(x)$

The function $\Upsilon(x)$ was introduced in [12] and can be defined by

$$\log\Upsilon(x) = \int_0^\infty \frac{dt}{t}\left[\left(\frac{Q}{2} - x\right)^2 e^{-t} + \frac{\sinh^2(\frac{Q}{2}-x)\frac{t}{2}}{\sinh\frac{bt}{2}\sinh\frac{t}{2b}}\right]. \tag{A.6}$$

The integral converges in the strip $0 < \mathrm{Re}(x) < Q$. For other x it is defined by the relations

$$\Upsilon(x+b) = b^{1-2bx}\gamma(bx)\Upsilon(x) \qquad \Upsilon(x+1/b) = b^{-1+2x/b}\gamma(x/b)\Upsilon(x). \tag{A.7}$$

From these last equations it follows that $\Upsilon(x)$ has zeros at

$$x = (m+1)b + \frac{(n+1)}{b}, \tag{A.8}$$

$$x = -mb - \frac{n}{b}, \tag{A.9}$$

for m, n non-negative integers.

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